Factorization and Painleve analysis of a class of nonlinear third-order partial differential equations

1995 J. Phys. A: Math. Gen. 28 2871

(http://iopscience.iop.org/0305-4470/28/10/017)

J. Phys. A: Math. Gen. 28 (1995) 2871–2888. Printed in the UK

Factorization and Painlevé analysis of a class of nonlinear third-order partial differential equations

Claire Gilson† and Andrew Pickering‡

† Department of Mathematics, University Gardens, University of Glasgow, Glasgow G12 8QW, Scotland, UK

‡ Dienst Theoretische Natuurkunde, Vrije Universiteit Brussel, 1050 Brussels, Belgium

Received 9 November 1994

Abstract. A class of fully nonlinear third-order partial differential equations (PDEs) is considered. This class contains several examples which have recently appeared in the literature and for which rather unusual travelling-wave solutions have been given. These solutions consist essentially of sums of exponentials; we trace the occurrence of these exponentials back to the existence of a *linear* subequation which appears as a factor in the travelling-wave reduction. In addition, we consider the Painlevé analysis of this set of equations, both for the original PDE and also for reductions to ordinary differential equations (ODEs). No equation in the class considered survives the combination of PDE and ODE tests. Also, an equation in the class considered which is known to be integrable is shown to possess only the 'weak Painlevé' property. Our analysis, therefore, confirms the limitations of the Painlevé test as a test for complete integrability when applied to fully nonlinear PDEs.

1. Introduction

We consider the class of fully nonlinear third-order partial differential equations (PDEs)

$$UU_{xxx} + \beta U_x U_{xx} - p^2(\beta+1)UU_x = U_t - \epsilon U_{xxt} + 2\kappa U_x \tag{1}$$

or, equivalently,

$$(U\partial + \beta U_x)(U_{xx} - p^2 U) = U_t - \epsilon U_{xxt} + 2\kappa U_x \tag{2}$$

where $\partial = \partial/\partial x$. Examples of equations of this form which have been discussed recently are, up to some rescalings, the Camassa–Holm (CH) equation [1, 2]

$$UU_{xxx} + 2U_x U_{xx} - 3UU_x = U_t - U_{xxt} + 2\kappa U_x \tag{3}$$

the Fornberg–Whitham (FW) equation [3–5]

$$UU_{xxx} + 3U_x U_{xx} - UU_x = U_t - U_{xxt} + U_x \tag{4}$$

and the Rosenau–Hyman (RH) equation [6]

$$UU_{xxx} + 3U_x U_{xx} + UU_x = U_t. \tag{5}$$

These correspond respectively to the choices of parameter values:

CH:	$\epsilon = 1$	$p^2 = 1$	$\beta = 2$	
FW:	$\epsilon = 1$	$p^2 = \frac{1}{4}$	$\beta = 3$	$\kappa = \frac{1}{2}$
RH:	$\epsilon = 0$	$p^2 = -\frac{1}{4}$	$\beta = 3$	$\kappa = 0.$

A Lax pair and bi-Hamiltonian structure have been given for the CH equation [1], and so this equation is assumed to be completely integrable. The FW equation was introduced in order to discuss qualitative features of wave breaking and it admits a wave of greatest height [3]. The RH equation arose in the study of the effect of nonlinear dispersion in the formation of patterns in liquid drops [6].

Rather unusual travelling-wave solutions of these equations have been given and their interaction properties analysed. For CH, in the case $\kappa = 0$, the so-called 'peakon'

$$U = c\mathrm{e}^{-|x-ct|} \tag{6}$$

has been given as a solution [1]. The interaction of such solutions is discussed in [2]. For FW, the wave of greatest height arises as a peaked limiting form of the travelling-wave solution [5]. When FW is written in the above form, this limiting case is

$$U = \tfrac{4}{3}\mathrm{e}^{-\frac{1}{2}|x-\frac{4}{3}t|}. \tag{7}$$

In the case of RH, the 'compacton' solution

$$U = \begin{cases} -\frac{8}{3}c\cos^2(\frac{1}{4}(x-ct)) & |x-ct| \leqslant 2\pi \\ 0 & |x-ct| > 2\pi \end{cases} \tag{8}$$

has been found [6]. These are solitary waves of finite wavelength, interactions of which produce a ripple of low amplitude compacton–anticompacton pairs [6].

All of the above solutions consist essentially of sums of exponentials; away from any discontinuities in their derivatives they are no more than the solutions of a linear equation. In this paper we locate this linear equation as a factor in the travelling-wave reduction. In fact for some examples, such as CH, we give a far more striking factorization which occurs at the PDE level.

We also consider the Painlevé analysis of equation (1). We find that the usual PDE test cannot be applied to (1) for certain ranges of parameter values because of problems in finding the dominant terms. We show how these problems can be overcome.

We then perform a Painlevé analysis of reductions of (1) to ordinary differential equations (ODEs). For these reductions, use is made of known classifications and also of the (classical) α-method. This analysis of ODE reductions proves more restrictive than does the PDE test.

Of course an ODE reduction cannot contain more information than the original PDE. For ODEs there is a properly defined Painlevé property, together with classifications of ODEs with this property, and a variety of methods providing necessary conditions. However, for PDEs, use is usually made of a single test only.

In addition we show that the integrable CH does not pass the Painlevé PDE test, but instead possesses only the 'weak Painlevé' property. However, FW is also shown to have this weak Painlevé property. Our analysis, therefore, confirms the limitations of the Painlevé test when applied to fully nonlinear PDEs.

The layout of this paper is as follows. Sections 2 and 3 deal with the factorization of our equation and with the integration of the travelling-wave reduction. Section 4 is concerned with the Painlevé analysis of (1). This begins with a description of the Painlevé tests, and then continues to consider the application of the Painlevé PDE test and of the Ablowitz–Ramani–Segur (ARS) Painlevé test. We are able to give a complete list of all equations of the form (1) which have travelling-wave reductions which have the Painlevé property; for these examples we then consider further ODE reductions. No equation in the class considered survives our Painlevé analysis. Finally, section 5 is devoted to a discussion and conclusions.

2. Solutions from linear equations: factorization

We begin by rewriting our equation (1) as

$$(U\partial + \beta U_x + \epsilon\partial_t)(U_{xx} - p^2U + \delta) - (1 - \epsilon p^2)U_t - (2\kappa + \beta\delta)U_x = 0 \tag{9}$$

where $\partial_t = \partial/\partial t$ and δ is an arbitrary constant.

We then see that for $\epsilon p^2 = 1$ our PDE admits a 'factorization' as a differential operator acting on a linear equation

$$(U\partial + \beta U_x + \epsilon\partial_t)(U_{xx} - p^2U + \delta) = 0 \tag{10}$$

provided that

$$2\kappa + \beta\delta = 0. \tag{11}$$

Thus, for any non-zero β we can always find a δ such that the PDE (1) can be written as in (10).

There is a similar factorization for the travelling-wave reduction of (1) which does not require $\epsilon p^2 = 1$. Putting

$$U(x, t) = f(\xi) \qquad \xi = x - ct \tag{12}$$

we obtain from (9)

$$\left[(f - \epsilon c)\frac{\mathrm{d}}{\mathrm{d}\xi} + \beta f_\xi\right](f_{\xi\xi} - p^2 f + \delta) + [c(1 - \epsilon p^2) - 2\kappa - \beta\delta]f_\xi = 0 \tag{13}$$

where $f_\xi = \mathrm{d}f/\mathrm{d}\xi$. So we see that the travelling-wave reduction factorizes as

$$\left[(f - \epsilon c)\frac{\mathrm{d}}{\mathrm{d}\xi} + \beta f_\xi\right](f_{\xi\xi} - p^2 f + \delta) = 0 \tag{14}$$

provided that

$$c(1 - \epsilon p^2) - 2\kappa - \beta\delta = 0. \tag{15}$$

So for any non-zero β, and for any choice of ϵ and p^2, we can always find a δ such that the travelling-wave reduction of (1) can be written in the form (14).

Assuming $p \neq 0$ we then find the following solutions of (1).

Case 1. $\epsilon p^2 = 1$ and $2\kappa + \beta\delta = 0$

$$U = a(t)\mathrm{e}^{-px} + b(t)\mathrm{e}^{px} + \frac{\delta}{p^2} \tag{16}$$

where $a(t)$ and $b(t)$ are arbitrary functions of t.

Case 2. $c(1 - \epsilon p^2) - (2\kappa + \beta\delta) = 0$

$$U = A\mathrm{e}^{-p\xi} + B\mathrm{e}^{p\xi} + \frac{\delta}{p^2} \qquad \xi = x - ct \tag{17}$$

where A and B are arbitrary constants.

For any non-zero β the above 'factorizations' can always be performed. Note that for $\epsilon p^2 \neq 1$ it is by including the constant δ that we are able to recover waves of different speeds c. Of course, (10) and (14) have more general solutions than (16) or (17), i.e. solutions of the full equation and not just of the linear subequation.

The above solutions are non-trivial, although in general they will be unphysical. However, for p purely imaginary—for example RH—they are bounded (and of course periodic).

2.1. Examples

In the case of CH we have from (10)

$$(U\partial + 2U_x + \partial_t)(U_{xx} - U - \kappa) = 0 \tag{18}$$

and from (16) the corresponding solution

$$U = a(t)\mathrm{e}^{-x} + b(t)\mathrm{e}^{x} - \kappa. \tag{19}$$

If we take $a(t) = A\mathrm{e}^{ct}$ and $b(t) = B\mathrm{e}^{-ct}$ then this becomes

$$U = A\mathrm{e}^{-\xi} + B\mathrm{e}^{\xi} - \kappa \tag{20}$$

(which may also be obtained from (17)). The peakon (6) can be understood to be a composite of the two exponentials which appear in (20). The structure of (6) guarantees zero boundary conditions. However, the price paid for this is a discontinuous first derivative. Consequently, the 'peakon' (6) is *not* a strong solution of CH, whereas (19)—for any $a(t)$, $b(t)$—is.

For FW and RH we have the factorization (14) of the travelling-wave reduction. For FW we obtain from (17) the solution

$$U = A\mathrm{e}^{-\frac{1}{2}\xi} + B\mathrm{e}^{\frac{1}{2}\xi} + c - \tfrac{4}{3}. \tag{21}$$

The peaked limit (7) can be understood to be a composite of the two exponentials which appear in (21). In fact, both (6) and (7) are properly understood as limiting cases of travelling-wave solutions under an assumption of zero boundary conditions. We say more about this in the next section.

If we now consider RH, we obtain from (17) the solution

$$U = \bar{A}\sin(\tfrac{1}{2}\xi) + \bar{B}\cos(\tfrac{1}{2}\xi) - \tfrac{4}{3}c \tag{22}$$

where $\bar{A}$, $\bar{B}$ are arbitrary constants.

The choice $\bar{A} = 0$, $\bar{B} = -\frac{4}{3}c$ then gives

$$U = -\tfrac{8}{3}c\cos^2(\tfrac{1}{4}\xi). \tag{23}$$

The compacton (8) consists of a single lump of this function, and remains a strong solution of RH since the latter can be written as [6]

$$(\tfrac{1}{2}U^2)_{xxx} + (\tfrac{1}{2}U^2)_x = U_t \tag{24}$$

and (8) is such that U^2 has everywhere three continuous derivatives. This compacton is not, of course, a solution of the linear subequation occurring in (14).

Away from discontinuities in their derivatives, the rather unusual travelling-wave solutions previously given to equations of the form (1) are no more than the solutions of a linear equation. We have shown that the linear equation arises as a 'factor' in the travelling-wave reduction.

3. Travelling-wave reduction

The ODE obtained from the travelling-wave reduction (12) may be integrated to yield

$$(f - \epsilon c)f_{\xi\xi} + \tfrac{1}{2}(\beta - 1)f_\xi^2 - \tfrac{1}{2}p^2(\beta + 1)f^2 + (c - 2\kappa)f + A = 0 \tag{25}$$

where A is a constant of integration. Equivalently,

$$FF_{\xi\xi} = -\tfrac{1}{2}(\beta - 1)F_\xi^2 + A_2F^2 + A_1F + A_0 \tag{26}$$

where $F = f - \epsilon c$ and

$$A_2 = \tfrac{1}{2}p^2(\beta+1) \tag{27}$$

$$A_1 = p^2(\beta+1)\epsilon c - (c - 2\kappa) \tag{28}$$

$$A_0 = \tfrac{1}{2}p^2(\beta+1)\epsilon^2c^2 - (c-2\kappa)\epsilon c - A. \tag{29}$$

This then gives, for $\beta \neq -1, 0, 1$,

$$F_\xi^2 = \frac{2}{\beta+1}A_2F^2 + \frac{2}{\beta}A_1F + \frac{2}{\beta-1}A_0 + BF^{1-\beta} \tag{30}$$

where B is a second constant of integration. For the cases $\beta = -1, 0$ and 1 we obtain

$$F_\xi^2 = -2A_1F - A_0 + BF^2 \tag{31}$$

$$F_\xi^2 = 2A_2F^2 + 2A_1F\log(F) - 2A_0 + BF \tag{32}$$

$$F_\xi^2 = A_2F^2 + 2A_1F + 2A_0\log(F) + B \tag{33}$$

respectively, where each A_i is to be evaluated at the particular choice of β, and we have used in (31) the fact that $A_2 = 0$ for $\beta = -1$. The solution of the travelling-wave reduction can, therefore, always be reduced to a quadrature.

3.1. *Limiting solutions*

Let us now consider the travelling-wave reductions of CH and FW, i.e.

$$(f-c)f_\xi^2 = f^2(f-c+2\kappa) - 2Af + D \tag{34}$$

$$D = B + 2Ac - 2\kappa c^2 \tag{35}$$

and

$$(f-c)^2f_\xi^2 = \tfrac{1}{4}f^2\left((f-c)^2 - (2f-3c)(c-\tfrac{4}{3})\right) - Af^2 + 2Acf + D \tag{36}$$

$$D = B - Ac^2 + \tfrac{1}{3}c^3 - \tfrac{1}{4}c^4 \tag{37}$$

respectively. For both of the above, an assumption that f and its derivatives vanish as $\xi \to \pm\infty$ gives $A = D = 0$.

To obtain the 'peakon' (6) one takes the solution of (34) in the case $A = D = 0$, and lets $\kappa \to 0$ [2]. The simplest way to see this is to note that when $A = D = 0$ and $\kappa = 0$, equation (34) becomes

$$(f-c)(f_\xi^2 - f^2) = 0 \tag{38}$$

and that one can take as a solution a composite of two exponentials provided that at the discontinuity in the derivative we take the amplitude $f(0) = c$.

Similarly for the peaked limiting solution (7) of FW, one takes the solution of (36) in the case $A = D = 0$, and lets $c \to 4/3$ [5]. For $A = D = 0$ and $c = 4/3$ equation (36) becomes

$$(f-c)^2(f_\xi^2 - \tfrac{1}{4}f^2) = 0 \tag{39}$$

and again, one can take as a solution a composite of two exponentials provided that at the discontinuity in the derivative we take the amplitude $f(0) = c = 4/3$.

In this way, it is easy to see the relationship between the solutions (20) and (21), and the limiting cases of the travelling-wave solutions (6) and (7).

3.2. Galilean invariance and further reductions

The condition for (1) to remain invariant under the change of variables

$$U(x,t) = \bar{U}(\bar{x},\bar{t}\,) + \epsilon\gamma \qquad \bar{x} = x - \gamma t \qquad \bar{t} = t \tag{40}$$

is

$$p^2(\beta+1)\epsilon - 1 = 0. \tag{41}$$

When the transformation (40) does not leave equation (1) invariant, it can be used to remove the term $2\kappa U_x$ from the right-hand side of (1). This is done by choosing γ to be a solution of

$$[p^2(\beta+1)\epsilon - 1]\gamma + 2\kappa = 0. \tag{42}$$

Thus, for example, in any equation of the form (1) with $\epsilon = 0$ we may set $\kappa = 0$, if this is not already the case.

For our three examples, CH and RH are not Galilean invariant, but FW is. Thus for CH there is a one-to-one correspondence between the solutions for $\kappa = 0$ and the solutions for $\kappa \neq 0$, although this correspondence involves a change in boundary conditions.

A consequence of having $\kappa = 0$ in (1) is that this equation then admits the scaling symmetry

$$U \to \lambda U \qquad t \to \lambda^{-1} t. \tag{43}$$

Thus, for any equation (1) with $\kappa = 0$ (or for which because of the lack of Galilean invariance can be transformed onto such an equation) we have in addition to the usual translation symmetry, which leads to the travelling-wave reduction, the symmetry (43), which then gives the reduction

$$U(x,t) = \frac{V(x)}{t}. \tag{44}$$

When equation (1) is Galilean invariant (and for any value of κ) we also obtain, using the direct method of Clarkson and Kruskal [7], the reduction

$$U(x,t) = -\epsilon(2at+b) + W(z) \qquad z = x + (at^2 + bt + d) \tag{45}$$

where a, b and d are constants. Thus, in addition to the travelling-wave reduction we always have at least one other reduction.

4. Painlevé analysis

4.1. The Painlevé tests

Before turning to the Painlevé analysis of equation (1), we first give a description of the various Painlevé tests, and of their relationship to the complete integrability of a PDE.

The connection between complete integrability and the Painlevé property was first noted by Ablowitz and Segur [8], who observed that similarity reductions of nonlinear PDEs solvable by an inverse scattering transform (IST) gave rise to nonlinear ODEs where the only movable singularities are poles. Joined by Ramani, they went on to formulate the Ablowitz–Ramani–Segur (ARS) conjecture [9, 10] (see also Hastings and McLeod [11]):

> Every ODE obtained as a similarity reduction of a completely integrable PDE is of P-type, perhaps after a change of variables.

It is this conjecture, taken here at face value, which forms the basis of the ARS Painlevé test, this being a test of PDEs for complete integrability. ARS defined an ODE to be of 'P-type' when the only movable singularities of any of its solutions are poles; a singularity is movable when its location depends on initial conditions (i.e. on constants of integration).

However, recent advances—in particular the recovery of a Lax pair [12], though of an unusual type, for an ODE with a movable natural boundary [13–15]—suggest that single-valued movable singularities other than poles should be allowed. This then leads us back to the original Painlevé property [16, 17]:

> An ODE has the Painlevé property when its general solution is free of movable branched singularities.

It is worth remarking here that Painlevé did allow unbranched movable essential singularities, and that one of the second-order equations with the Painlevé property that he lists in [17] does have such a singularity. Of course, if we concentrate on the general solution, asking that ODE reductions of PDEs have the Painlevé property rather than just be of P-type, simply broadens the class of equations to be considered.

Thus, when applying the ARS Painlevé test, we check all similarity reductions to ODEs for the Painlevé property. For certain classes of equations, classifications of ODEs with the Painlevé property exist. If our reduction falls within such a class, then we can check to see whether it occurs in the corresponding classification. If the equation does not fall into a class for which such a classification exists, then we have to resort to other methods. One such method is the ARS algorithm [9, 10], which, after Kowalevski [18, 19], seeks a solution as an expansion about a movable pole. Other methods by which we may test an ODE for the Painlevé property are the so-called α-method, due to Painlevé [16], and the method of Bureau [20]. All of these methods provide *necessary* conditions for an ODE to have the Painlevé property.

In order to overcome the need to obtain all the similarity reductions of a PDE to ODEs, Weiss, Tabor and Carnevale (WTC) proposed a test of the single-valuedness of solutions of a PDE which could be applied directly to the PDE [21]. This constituted a direct extension of the ARS algorithm. However, in going from ODEs to PDEs there are two observations that must be taken into account. The first of these is that the main difference between an analytic function of several complex variables $x_1, \ldots, x_s$, and an analytic function of one complex variable, is that its non-removable singularities are not isolated; instead they occur upon analytic manifolds of (real) dimension $2s-2$ [22]. The second observation is that *characteristic singular manifolds must be excluded from consideration, since even linear* PDEs may exhibit branching along characteristics [23]. Necessary conditions for a PDE to have the Painlevé property are then provided by the WTC Painlevé test [21].

Given a PDE, for example in $U = U(x,t)$, this test consists of seeking a solution as an expansion

$$U = \Phi^{\rho}(x,t) \sum_{j=0}^{\infty} U_j(x,t)\Phi^j(x,t) \tag{46}$$

in the neighbourhood of a non-characteristic movable singular manifold $\Phi(x,t) = 0$.

Such an analysis first requires a choice of expansion family. This is a choice of leading-order exponent ρ, leading-order coefficient U_0, and dominant terms $\hat{K}[U]$. For each family there is a set of indices $\mathcal{R} = \{r_1, \ldots, r_N\}$ which give the values of j for which arbitrary data should be introduced in (46). Following the terminology introduced in [9, 10], these indices are often referred to as 'resonances'. A choice of family is, therefore, given as

$$\rho \qquad U_0 \qquad \hat{K}[U] \qquad \sigma \qquad \mathcal{R} = \{r_1, \ldots, r_N\} \tag{47}$$

where σ is the singularity-order of $\hat{K}[U]$ when U is given by (46).

A perturbative extension of the WTC test [24] allows for each family the construction of a solution with arbitrary data corresponding to *every* index. This then gives the following necessary conditions for a PDE to have the Painlevé property. For any family which represents either the general or a particular solution ρ must be integer; the indices must be distinct integers; and all compatibility conditions corresponding to each index must be satisfied.

For any 'maximal' family (when the number of indices N is the same as the order of the equation) this perturbative analysis gives a local representation of the general solution. We have a program [25] with which to carry out a generalized version of this perturbative test; this was used to carry out all such calculations presented herein.

4.2. *Application of the Painlevé PDE test*

We begin by considering the application of the Painlevé PDE test to (1). In doing so we use in (46) the ansatz [26], i.e.

$$\Phi(x,t) = \phi(x,t) = x + \psi(t) \qquad U_j(x,t) = U_j(t) \tag{48}$$

to simplify the calculations. The characteristics of (1) are given by

$$x = X(\tau) \qquad t = T(\tau) \qquad \left(\frac{\mathrm{d}T}{\mathrm{d}\tau}\right)^2 \left(U\frac{\mathrm{d}T}{\mathrm{d}\tau} - \epsilon\frac{\mathrm{d}X}{\mathrm{d}\tau}\right) = 0 \tag{49}$$

and so (1) has $\Phi(x,t) = t - t_0 = 0$ as, at least, a double characteristic. The ansatz (48), therefore, explicitly excludes such characteristic manifolds from being considered as the singular manifold about which the expansion (46) is made; remember that characteristic singularities are to be disregarded when determining whether or not a PDE has the Painlevé property. This also means that although the arbitrary functions $a(t)$ and $b(t)$ in (19) can be assigned whatever singularity structure we like—including branching—this is not sufficient to claim that CH does not have the Painlevé property.

We now turn to the determination of possible expansion families (47). Since we are interested primarily in maximal families, we make the requirement that at least one of the third-order derivatives in (1) contributes to the dominant terms $\hat{K}[U]$. We also ask that the resulting expansion is not simply Taylor, or a special case thereof. This then gives us two possible families.

The first of these is

$$\begin{aligned} &\rho = \frac{2}{\beta+1} \qquad U_0 \qquad \hat{K}[U] = UU_{xxx} + \beta U_x U_{xx} \\ &\sigma = -\frac{3\beta-1}{\beta+1} \qquad \mathcal{R} = \left\{-1, 0, 2\frac{\beta-1}{\beta+1}\right\} \end{aligned} \tag{50}$$

(U_0 is arbitrary), for which the following dominance conditions must hold:

$$\epsilon = 0 \qquad \beta < -1 \text{ or } 1 \neq \beta > 0 \tag{51}$$

$$\epsilon \neq 0 \qquad \beta < -1. \tag{52}$$

These conditions arise from the requirement that the terms $\hat{K}[U]$ actually be dominant. They are already dominant on the left-hand side of (1), so we only need to worry about the right-hand side. In the case $\epsilon = 0$ this means that they must dominate over U_t and $2\kappa U_x$, and in the case $\epsilon \neq 0$ they must dominate over $-\epsilon U_{xxt}$.

In the special case $\epsilon = 0$ we also have the second family

$$\begin{aligned} &\rho = 2 \qquad U_0 = \frac{\psi' + 2\kappa}{2\beta} \qquad \hat{K}[U] = UU_{xxx} + \beta U_x U_{xx} - U_t - 2\kappa U_x \\ &\sigma = 1 \qquad \mathcal{R} = \{-2, -1, -2\beta\}. \end{aligned} \tag{53}$$

Here, the dominant terms given are those which contribute to the determination of the indices $\mathcal{R}$; it is, however, only the terms $\beta U_x U_{xx} - U_t - 2\kappa U_x$ that determine U_0. In what follows, little use is made of this second family, since it exists only for the case $\epsilon = 0$. Instead, we concentrate on the family (50).

From our leading-order analysis for the family (50) we can conclude immediately that when $\epsilon = 0$, for any $1 \neq \beta > 0$, our equation does not have the Painlevé property because we have a maximal family with non-integer leading-order exponent. Thus, for example, RH has the family

$$\rho = \tfrac{1}{2} \qquad U_0 \qquad \hat{K}[U] = UU_{xxx} + 3U_x U_{xx} \qquad \sigma = -2 \qquad \mathcal{R} = \{-1, 0, 1\} \tag{54}$$

and so does not pass the Painlevé PDE test. However, RH does admit a so-called 'weak Painlevé' [27] expansion

$$U = \phi^{\frac{1}{2}}(x, t) \sum_{j=0}^{\infty} U_{j/2}(t) \phi^{j/2}(x, t) \tag{55}$$

where U_0 and U_1 are arbitrary. Unfortunately this tells us little about the integrability or otherwise of RH, since the (integrable) Dym equation and the (non-integrable) 'cubic KdV' equation both admit such weak Painlevé expansions [28, 29]. However, if we consider the second family (53) of RH (note that all indices are negative),

$$\begin{aligned} &\rho = 2 \qquad U_0 = \frac{\psi'}{6} \qquad \hat{K}[U] = UU_{xxx} + 3U_x U_{xx} - U_t \\ &\sigma = 1 \qquad \mathcal{R} = \{-6, -2, -1\} \end{aligned} \tag{56}$$

we find, using the extended test developed in [24], failed compatibility conditions at first-order of perturbation. Thus RH also exhibits logarithmic branching, which is a stronger indication of non-integrability.

In the case $\epsilon \neq 0$ the condition for dominance (52) is much more restrictive and means, for example, that for CH and FW ($\beta = 2$ and $\beta = 3$ respectively) we are unable to build a Painlevé expansion of the form (46) since the terms $\hat{K}[U]$ are not dominant. However, it is possible—in a manner analogous to that used in the weak Painlevé analysis of ODEs, see for example [30]—to overcome this problem and increase the range of values of β for which the PDE test can be applied.

In (1) we set $U = V - \epsilon\psi'(t)$ to obtain

$$\begin{aligned} VV_{xxx} + \beta V_x V_{xx} - p^2(\beta + 1)VV_x = V_t - \epsilon(V_{xxt} - \psi' V_{xxx}) \\ + (2\kappa - p^2(\beta + 1)\epsilon\psi')V_x - \epsilon\psi''. \end{aligned} \tag{57}$$

We now consider a leading-order analysis of this equation for the same dominant terms $\hat{K}[V] = VV_{xxx} + \beta V_x V_{xx}$. When V has leading-order exponent ρ, the leading-order exponent of the combination $(V_{xxt} - \psi' V_{xxx})$ is in fact $\rho - 2$ and not $\rho - 3$. Therefore when seeking a Painlevé expansion for V in the case $\epsilon \neq 0$ we obtain different conditions for $\hat{K}[V]$ to dominate over the right-hand side:

$$\epsilon \neq 0 \qquad \beta < -1 \text{ or } \beta > 1. \tag{58}$$

This means that for equation (1) with $\epsilon \neq 0$ we can seek a Painlevé expansion in the form

$$U = -\epsilon\psi'(t) + \phi^{\rho}(x,t)\sum_{j=0}^{\infty} U_j(t)\phi^j(x,t) \tag{59}$$

having the conditions for dominance (58). For $\beta < -1$ we already know that we do not need to include the extra term $-\epsilon\psi'$; indeed in this case, since $\rho < 0$, this term can be absorbed into the sum in (59). It is for $\beta > 1$ that the above modification is useful; since then $\rho > 0$, what we have done is to include an extra lower-order term in the expansion.

This then allows us to conclude that for $\epsilon \neq 0$ and any $\beta > 1$—and in particular for CH and FW—our equation does not have the Painlevé property. However, both CH and FW do admit weak Painlevé expansions which are, respectively,

$$U = -\psi'(t) + \phi^{2/3}(x,t)\sum_{j=0}^{\infty} U_{j/3}(t)\phi^{j/3}(x,t) \tag{60}$$

where U_0 and $U_{2/3}$ are arbitrary and

$$U = -\psi'(t) + \phi^{1/2}(x,t)\sum_{j=0}^{\infty} U_{j/2}(t)\phi^{j/2}(x,t) \tag{61}$$

where U_0 and U_1 are arbitrary. Since CH is integrable, and so might be expected to have the full Painlevé property, our analysis confirms the limitations of the Painlevé test when applied to fully nonlinear PDEs.

For $\beta < -1$ we can apply the PDE test for the family (50), without including an extra term in the expansion, for any value of ϵ. Asking that the leading-order exponent ρ be integer N ($\neq 0$), so $\beta = 2/N - 1$, we can rewrite the family (50) as

$$\begin{aligned} &\rho = N \qquad U_0 \qquad \hat{K}[U] = UU_{xxx} - \left(1 - \frac{2}{N}\right)U_xU_{xx} \\ &\sigma = 2N - 3 \qquad \mathcal{R} = \{-1, 0, 2 - 2N\}. \end{aligned} \tag{62}$$

We have applied the PDE test for this family to equation (1) with $\beta = 2/N - 1$, $N = -1, \ldots, -10$. We found that when such N are even, the corresponding equation passes the PDE test, whereas for $N = -(2n+1)$, $n = 0, \ldots, 4$ the compatibility condition at $r = 4(n+1)$ requires

$$p^{2n}U_{0t} = 0. \tag{63}$$

All compatibility conditions are required to be satisfied without any restriction being placed on the arbitrary function U_0. Thus for $N = -1$ ($n = 0$) the equation fails the PDE test, whereas for $N = -3, -5, -7, -9$ we must have $p = 0$. We expect this pattern of passing for N even, or requiring $p = 0$ for N odd, to continue for other negative integer leading-orders.

4.3. Application of the Painlevé ODE test

In this section we consider the application of the ARS Painlevé test for complete integrability, and so are interested in the Painlevé property of reductions to ODEs. We begin with the travelling-wave reduction

$$U(x,t) = F(\xi) + \epsilon c \qquad \xi = x - ct \tag{64}$$

which, we recall, gives

$$FF_{\xi\xi} = -\tfrac{1}{2}(\beta - 1)F_\xi^2 + A_2F^2 + A_1F + A_0 \tag{65}$$

with A_2, A_1 and A_0 given by (27), (28) and (29) earlier.

We prefer to consider a Painlevé analysis of equation (65) rather than of the first-order equation (30), since the former has polynomial form. Whilst it is true that if the right-hand side of (30) is rational, then in order to have the Painlevé property it must in fact be a polynomial of degree not exceeding 4 (see for example [31] pp 418–9), to use this result requires an assumption that β is integer. There are equations of the form (30) with solutions free of movable branch points but β rational, for example $A_0 = 0$ and $\beta = -1/2$.

Remark. This result, that for (30) to have the Painlevé property when β is integer, the left-hand side must be polynomial of degree $\leqslant 4$, excludes the consideration of limiting solutions such as (6) and (7), since in Painlevé analysis we deal only with irreducible equations. In fact, these limiting solutions of the form $e^{-\nu|z|}$, $\nu \neq 0$, although perfectly single-valued, are *nowhere differentiable* with respect to complex z.

Another reason for using (65) rather than (30) is that it falls into the class of second-order ODEs studied by Painlevé and Gambier [16, 17, 32], and later by Bureau [33]. We can, therefore, use these classifications to give all ODEs of the form (65) which have the Painlevé property. Throughout this section we also make use of the α-method.

We find four PDEs which have travelling-wave reductions that have the Painlevé property. One of these fails the Painlevé PDE test; for the remaining three we are able to undertake a further analysis using the reductions (44) and (45). Again, use is made of the α-method, this being more generally applicable than the ARS algorithm.

In equation (65) it should be noted that A_0 depends on the arbitrary constant of integration A, and so if we require that the *general solution* of the third-order ODE resulting from the travelling-wave reduction has the Painlevé property (i.e. for arbitrary A), then we must insist that A_0 is entirely arbitrary. Therefore, when looking for equations of the form (65) with the Painlevé property we assume that A_0 is non-zero, since otherwise such equations correspond only to the classes of particular solutions of the travelling-wave reduction. We make this assumption $A_0 \neq 0$ in all that follows. Note also that a constraint on A_0—for example $A_0 = 0$—does not provide us with any information about the parameters in our original PDE.

4.3.1. Travelling-wave reductions with the Painlevé property We begin by considering the application of the α-method to equation (65). This means making a change of variables depending on a parameter α such that the resulting system is equivalent to the original for all $\alpha \neq 0$, and such that the resulting expression for $F_{\xi\xi}$ is analytic at $\alpha = 0$. A solution of the resulting ODE is then sought as a Taylor series in α; the necessary conditions for the Painlevé property are obtained by requiring single-valuedness of each coefficient in this Taylor series. In particular, the solution of the 'simplified equation', obtained by setting $\alpha = 0$ in the transformed equation, must be single-valued.

In (65) we make the change of variables

$$\xi = \zeta_0 + \alpha\zeta \tag{66}$$

to obtain

$$FF_{\zeta\zeta} = -\tfrac{1}{2}(\beta - 1)F_\zeta^2 + \alpha^2(A_2F^2 + A_1F + A_0). \tag{67}$$

Then, seeking a solution of (67) as

$$F = \sum_{j=0}^{\infty} F_j(\zeta)\alpha^j \tag{68}$$

we obtain for F_0 the simplified equation

$$F_0 F_{0\zeta\zeta} = -\tfrac{1}{2}(\beta - 1)F_{0\zeta}^2 \tag{69}$$

which has the general solution

$$\beta = -1 \qquad F_0(\zeta) = Ce^{D\zeta} \tag{70}$$

$$\beta \neq -1 \qquad F_0(\zeta) = C(\zeta + D)^{\frac{2}{\beta+1}} \tag{71}$$

where C and D are arbitrary constants.

We thus have the first necessary conditions [16] for (65) to have the Painlevé property: either $\beta = -1$ or $2/(\beta + 1) = N$ for some non-zero integer N. This then means that for any $1 \neq \beta > 0$, and for any value of ϵ, the travelling-wave reduction of (1) does not have the Painlevé property. This is, therefore, true of our three examples CH, FW and RH (the travelling-wave reduction of FW has previously been considered by McLeod and Olver [34]). Of course, for these three examples, we have already shown the existence of such branching at the PDE level.

Let us now continue to determine which equations of the form (65)—with $A_0 \neq 0$—do have the Painlevé property. In doing so we will make use of the known classifications of second-order ODEs [16, 17, 32, 33]. The results of the Painlevé–Gambier classification may be found in Ince [35], although with many errors and omissions now corrected by Cosgrove [36] (useful comments can also be found in [37]). Some care has to be taken in making use of these classifcations, since they are classifications of *canonical* equations, i.e. up to transformations under the Möbius group

$$F(\xi) = \frac{\lambda(\xi)G(\eta) + \lambda_0(\xi)}{\mu(\xi)G(\eta) + \mu_0(\xi)} \qquad \eta = \varphi(\xi) \tag{72}$$

where $\lambda(\xi)\mu_0(\xi) - \mu(\xi)\lambda_0(\xi) \neq 0$ and $\varphi(\xi)$ is non-constant.

For $\beta = -1$ we know from (31) that the travelling-wave reduction does have the Painlevé property (in fact the general solution of (31) is analytic). We therefore obtain the following.

Equation (1) with $\beta = -1$. The corresponding PDE is not Galilean invariant and so we may use the transformation (40) to set $\kappa = 0$. This then gives

$$UU_{xxx} - U_x U_{xx} = U_t - \epsilon U_{xxt}. \tag{73}$$

We now consider equation (65) for $\beta = 2/N - 1$ (N non-zero integer):

$$F_{\xi\xi} = \left(1 - \frac{1}{N}\right)\frac{F_\xi^2}{F} + A_2 F + A_1 + A_0\frac{1}{F}. \tag{74}$$

First we take the choices $N = \pm 1$ (equations of type I in the classification [35]).

For $N = 1$ ($\beta = 1$) we obtain

$$F_{\xi\xi} = A_2 F + A_1 + A_0\frac{1}{F}. \tag{75}$$

Employing the α-method with

$$F = \alpha H \qquad \xi = \zeta_0 + \alpha\zeta \tag{76}$$

we obtain the simplified equation

$$H_{\zeta\zeta} = A_0 \frac{1}{H}. \tag{77}$$

Further applications of the α-method show that a necessary condition for (77) to have the Painlevé property (as a second-order ODE) is $A_0 = 0$ (see [33] pp 247–8 or [16] pp 218–9). Since we are insisting that the constant of integration A is arbitrary, we discard this example.

For $N = -1$ ($\beta = -3$), we can see immediately from (30) that the travelling-wave reduction has the Painlevé property, since for this value of β the general solution of (30) is expressed in terms of elliptic functions. This is our second case.

Equation (2) with $\beta = -3$ ($N = -1$).

$$UU_{xxx} - 3U_xU_{xx} + 2p^2UU_x = U_t - \epsilon U_{xxt} + 2\kappa U_x. \tag{78}$$

However, as remarked earlier, this equation fails the Painlevé PDE test.

We now consider equation (74) for $N = \pm 2, \pm 3, \pm 4, \ldots$, i.e. equations of type III (the first complete discussion of which was given by Gambier [38, 39]). Using the lists of canonical equations of type III in [35, 36], we are able to complete our list of travelling-wave reductions with the Painlevé property; there are two further examples to be given. Remember that we are imposing the constraint $A_0 \neq 0$.

Our first example of an equation of type III is for $N = 2$ ($\beta = 0$) and $A_1 = 0$:

$$F_{\xi\xi} = \frac{1}{2}\frac{F_\xi^2}{F} + A_2F + A_0\frac{1}{F} \tag{79}$$

which for $A_0 \neq 0$ appears in the classification as

$$G_{\eta\eta} = \frac{1}{2}\frac{G_\eta^2}{G} - \frac{1}{2}\frac{1}{G}. \tag{80}$$

This is a particular case of equation (XXVII) in [35], labelled (XXVIIa) in [36]. In the case $N = 2$ it is equivalent to (XXXII) in [35]. Equations (79) and (80) are equivalent under a transformation of the form (72).

The extra constraint $A_1 = 0$ is in fact relatively simple to derive. Let us return to our application of the α-method to equation (65)—and so making the transformation (66) to obtain (67)—in the particular case $\beta = 0$. It is sufficient to take for F_0 and F_1 in (68) the particular solutions

$$F_0(\zeta) = (\zeta + D)^2 \qquad F_1(\zeta) = 0. \tag{81}$$

The equation for F_2 then reads

$$F_{2\zeta\zeta} - \frac{2}{\zeta + D}F_{2\zeta} + \frac{2}{(\zeta + D)^2}F_2 = A_2(\zeta + D)^2 + A_1 + \frac{A_0}{(\zeta + D)^2} \tag{82}$$

which has the particular integral

$$F_2(\zeta) = \tfrac{1}{6}A_2(\zeta + D)^4 + A_1(\zeta + D)^2[\log(\zeta + D) - 1] + \tfrac{1}{2}A_0. \tag{83}$$

It then follows that a necessary condition for the absence of movable logarithmic branching when $\beta = 0$ is that $A_1 = 0$; the sufficiency of this condition is clear from (32), which in fact has an analytic general solution when $A_1 = 0$.

Thus we have our third equation.

Equation (3) $\beta = 0$ ($N = 2$), $A_1 = 0$. The condition $A_1 = 0$ could be used to determine the wave speed c, but to have the travelling-wave reduction having the Painlevé property only for a fixed wave speed would not be very interesting. In fact, in this case—which requires $\epsilon p^2 \neq 1$—the PDE is not Galilean invariant and so can be transformed onto an equation having $\kappa = 0$; then we are left only with an equation with a stationary flow that has the Painlevé property. If we require that the condition $A_1 = 0$ holds for arbitrary wave speed then we must have $\epsilon p^2 = 1$ (Galilean invariance) and $\kappa = 0$. This then gives, for $\beta = 0$ ($N = 2$), $\epsilon p^2 = 1$, $\kappa = 0$,

$$UU_{xxx} - p^2 UU_x = U_t - \frac{1}{p^2} U_{xxt}. \tag{84}$$

Here we may rescale $p^2 = 1$.

Our second example of an equation of type III [35] is, for $N = -2$ ($\beta = -2$),

$$F_{\xi\xi} = \frac{3}{2}\frac{F_\xi^2}{F} + A_2 F + A_1 + A_0 \frac{1}{F} \tag{85}$$

for which when $A_0 \neq 0$ the corresponding canonical equation is

$$G_{\eta\eta} = \frac{1}{2}\frac{G_\eta^2}{G} + \tfrac{3}{2} G^3 + B_1 G^2 + B_2 G \tag{86}$$

a particular case of (XXX) in [35]. Equations (85) and (86) are equivalent under a transformation of the form (72), B_1 and B_2 being given in terms of A_0, A_1 and A_2. That (85) has the Painlevé property is easy to see from (30), which in the case $\beta = -2$ has a general solution expressible in terms of elliptic functions.

We then obtain our fourth example of a PDE of the form (1), the travelling-wave reduction of which has the Painlevé property.

Equation (4) with $\beta = -2$ ($N = -2$).

$$UU_{xxx} - 2U_x U_{xx} + p^2 UU_x = U_t - \epsilon U_{xxt} + 2\kappa U_x. \tag{87}$$

We recall that this equation also passes the Painlevé PDE test for the family (50).

4.3.2. Further analysis: other similarity reductions Of the four PDEs listed above two have $\kappa = 0$ and so we are also able to consider the reduction (44)

$$U(x,t) = \frac{V(x)}{t} \tag{88}$$

corresponding to the scaling symmetry (43).

For equation (73) this reduction gives the ODE

$$VV_{xxx} - V_x V_{xx} + V - \epsilon V_{xx} = 0. \tag{89}$$

Applying the α-method with

$$V = \alpha^{-3} W \qquad x = y_0 + \alpha^{-1} y \tag{90}$$

gives the simplified equation

$$WW_{yyy} - W_y W_{yy} + W = 0. \tag{91}$$

This has the family (note the positive leading-order exponent)

$$\begin{aligned} &\rho = 3 \qquad W_0 = \tfrac{1}{12} \qquad \hat{K}[W] = WW_{yyy} - W_y W_{yy} + W \\ &\sigma = 3 \qquad \mathcal{R} = \left\{-1, -1 \pm \sqrt{13}\right\} \end{aligned} \tag{92}$$

and so the reduction (89) does not have the Painlevé property (because of the non-integer indices of the family (92)).

For equation (84), after rescaling $p^2 = 1$, the reduction (88) gives the ODE

$$VV_{xxx} - VV_x + V - V_{xx} = 0. \tag{93}$$

Applying the α-method with

$$V = \alpha W \qquad x = y_0 + \alpha y \tag{94}$$

gives the simplified equation

$$WW_{yyy} - W_{yy} = 0 \tag{95}$$

which integrates to

$$WW_{yy} - \tfrac{1}{2}W_y^2 - W_y - E = 0. \tag{96}$$

Applying the α-method again with

$$W = \alpha^{-1}Z \tag{97}$$

gives

$$ZZ_{yy} - \tfrac{1}{2}Z_y^2 = \alpha Z_y + \alpha^2 E. \tag{98}$$

Seeking a solution as

$$Z = \sum_{j=0}^{\infty} Z_j(y)\alpha^j \tag{99}$$

we take for Z_0 the particular solution

$$Z_0(y) = (y + D)^2 \tag{100}$$

the equation for Z_1 then reads

$$Z_{1yy} - \frac{2}{y+D}Z_{1y} + \frac{2}{(y+D)^2}Z_1 = \frac{2}{y+D} \tag{101}$$

which has the particular integral

$$Z_1(y) = -2(y + D)\,(1 + \log(y + D))\,. \tag{102}$$

Thus (93) does not have the Painlevé property.

So (73) and (84) both fail the ARS Painlevé test. Since (78) fails the Painlevé PDE test, we are then left only with equation (87). It is this remaining equation that we now consider.

Since when (87) is not Galilean invariant we may set $\kappa = 0$, we need consider only the following two cases: (i) $\kappa = 0$ and (ii) $\kappa \neq 0$ and Galilean invariant, i.e. $\epsilon p^2 + 1 = 0$. We take these two cases in turn.

For the first case we can again study the reduction (88), which gives

$$VV_{xxx} - 2V_xV_{xx} + p^2VV_x + V - \epsilon V_{xx} = 0. \tag{103}$$

Applying the α-method with

$$V = \alpha W \qquad x = y_0 + \alpha y \tag{104}$$

gives the simplified equation

$$WW_{yyy} - 2W_yW_{yy} - \epsilon W_{yy} = 0 \tag{105}$$

which integrates to

$$WW_{yy} - \tfrac{3}{2}W_y^2 - \epsilon W_y + E = 0. \tag{106}$$

The transformation $W = 1/Z$ then brings this to the canonical form

$$Z_{yy} = \frac{1}{2}\frac{Z_y^2}{Z} + \epsilon Z Z_y + E Z^3. \tag{107}$$

The only case of the above in which the constant of integration E is allowed to be entirely arbitrary is when $\epsilon = 0$ (see [35], p 336). Thus we have $\kappa = \epsilon = 0$.

We can then consider the Painlevé PDE test for the second family (53):

$$\begin{aligned} &\rho = 2 \qquad U_0 = -\tfrac{1}{4}\psi' \qquad \hat{K}[U] = UU_{xxx} - 2U_xU_{xx} - U_t \\ &\sigma = 1 \qquad \mathcal{R} = \{-2, -1, 4\}. \end{aligned} \tag{108}$$

We find that the compatibility condition for the index at $j = 4$ is not identically satisfied for any p; thus the equation exhibits logarithmic branching.

We are then left with the second case of equation (87), i.e. $\kappa \neq 0$ and $\epsilon p^2 + 1 = 0$. We may rescale to set $\kappa = \frac{1}{2}$, $p^2 = 1$ and $\epsilon = -1$:

$$UU_{xxx} - 2U_xU_{xx} + UU_x = U_t + U_{xxt} + U_x. \tag{109}$$

Since this equation is Galilean invariant we can take the reduction (45)

$$U(x,t) = (2at + b) + W(z) \qquad z = x + (at^2 + bt + d) \tag{110}$$

to obtain

$$WW_{zzz} - 2W_zW_{zz} + WW_z = 2a + W_z. \tag{111}$$

For $a = 0$ the reduction (110) is just the travelling-wave reduction (64), which we know has the Painlevé property. We therefore consider equation (111) for $a \neq 0$.

The family corresponding to (50) passes the Painlevé test. However, in the case $a \neq 0$ we also have the two families defined by

$$\begin{aligned} &\rho = \tfrac{3}{2} \qquad \tfrac{21}{8}W_0^2 + 2a = 0 \qquad \hat{K}[W] = WW_{zzz} - 2W_zW_{zz} - 2a \\ &\sigma = 0 \qquad \mathcal{R} = \left\{-1, \tfrac{5\pm\sqrt{109}}{4}\right\}. \end{aligned} \tag{112}$$

Since in (112) we have both leading-order exponent and indices non-integer, it follows that if we take a reduction (110) with $a \neq 0$ we obtain an ODE which does not have the Painlevé property.

Thus our Painlevé analysis leads us to discard all equations of the form (1).

5. Conclusions

We noted that the limiting solutions to equations of the form (1) are, away from discontinuities in their derivatives, no more than the solutions of a linear equation. We have shown how this linear equation arises as a 'factor' in the travelling-wave reduction. For some examples we are able to give a far more striking factorization at the PDE level. These methods also allow us to write down classes of solutions to the equation considered.

In addition, we have considered the Painlevé analysis of (1). A combination of the PDE test and the ARS test of reductions to ODEs forces us to discard all equations in the class considered.

We also found that by including an extra lower-order term in the Painlevé expansion when the leading-order exponent is positive, we could extend the class of equations to which this test may be applied. In this way both CH and FW can be shown to possess only the 'weak Painlevé' property. Since CH is known to be integrable, this then confirms the limitations of the Painlevé test when applied to fully nonlinear PDEs. We should remember, of course, that those proofs that integrable PDEs have the Painlevé property which have been given—see for example [34]—make certain assumptions about the scattering problem involved, and also that it is for the functional $Q[U]$ recovered from inverse scattering that meromorphy has been shown, and then only for those solutions U satisfying conditions on initial data which allow the inverse scattering formalism to go through.

An alternative approach might be to ask whether it is possible to transform CH onto an equation which passes the Painlevé test. This is the approach advocated in [40]. Indeed such a transformation, onto the first negative flow of the KdV hierarchy, has been given [41]. However, in deriving this transformation use was made of the (spatial part of the) Lax pair of CH. One can, therefore, imagine that given *any* partial differential equation the question of whether such a transformation can be found may not always be quite so tractable. In practice, of course, one would like a method of testing any given equation directly. The question which remains to be answered is: how many other equations in the class (1) are actually integrable but in their Painlevé analysis exhibit similar behaviour to CH?

References

[1] Camassa R and Holm D D 1993 *Phys. Rev. Lett.* **71** 1661–64
[2] Camassa R, Holm D D and Hyman J M 1994 *Adv. Appl. Mech.* **31** 1–33
[3] Whitham G B 1967 *Proc. R. Soc.* A **299** 6–25
[4] Whitham G B 1974 *Linear and Nonlinear Waves* (New York: Wiley)
[5] Fornberg B and Whitham G B 1978 *Phil. Trans. R. Soc.* A **289** 373–404
[6] Rosenau P and Hyman J M 1993 *Phys. Rev. Lett.* **70** 564–7
[7] Clarkson P A and Kruskal M D 1989 *J. Math. Phys.* **30** 2201–13
[8] Ablowitz M J and Segur H 1977 *Phys. Rev. Lett.* **38** 1103–6
[9] Ablowitz M J, Ramani A and Segur H 1978 *Lett. Nuovo Cimento* **23** 333–8
[10] Ablowitz M J, Ramani A and Segur H 1980 *J. Math. Phys.* **21** 715–21
[11] Hastings S P and McLeod J B 1980 *Arch. Ration. Mech. Anal.* **73** 31–51
[12] Takhtajan L A 1992 *Teoret. Mat. Fiz.* **93** 330–41 (Engl. transl. 1992 *Theor. Math. Phys.* **93** 1308–17)
[13] Chazy J 1909 *C. R. Acad. Sci., Paris* **149** 563–5
[14] Chazy J 1910 *C. R. Acad. Sci., Paris* **150** 456–458
[15] Chazy J 1911 *Acta Math.* **34** 317–85
[16] Painlevé P 1900 *Bull. Soc. Math., France* **28** 201–61
[17] Painlevé P 1902 *Acta Math.* **25** 1–85
[18] Kowalevski S 1889 *Acta Math.* **12** 177–232
[19] Kowalevski S 1889 *Acta Math.* **14** 81–93
[20] Bureau F J 1939 *Acad. R. Belg. Bull. Cl. Sci.* **25** 51–68
[21] Weiss J, Tabor M, and Carnevale G 1983 *J. Math. Phys.* **24** 522–6
[22] Osgood W F 1914 *Topics in the Theory of Functions of Several Complex Variables (The Madison Colloquium 1913 Part II)* (Providence, RI: AMS) (Reprint 1966 (New York: Dover))
[23] Ward R S 1984 *Phys. Lett.* A**102** 279–82
[24] Conte R, Fordy A P and Pickering A 1993 *Physica* **69D** 33–58
[25] Pickering A 1994 Perturbative Painlevé analysis of nonlinear differential equations: a MATHEMATICA program *Preprint*
[26] Jimbo M, Kruskal M D and Miwa T 1982 *Phys. Lett.* **92A** 59–60
[27] Ramani A, Dorizzi B and Grammaticos B 1982 *Phys. Rev. Lett.* **49** 1539–41
[28] Weiss J 1983 *J. Math. Phys.* **24** 1405–13
[29] Weiss J 1986 *J. Math. Phys.* **27** 1293–305
[30] Ramani A, Grammaticos B and Bountis T 1989 *Phys. Rep.* **180** 159–245
[31] Hille E 1976 *Ordinary Differential Equations in the Complex Domain* (New York: Wiley)
[32] Gambier B 1910 *Acta Math.* **33** 1–55
[33] Bureau F J 1964 *Ann. Mat. Pura Appl.* **64** 229–364
[34] McLeod J B and Olver P J 1983 *SIAM J. Math. Anal.* **14** 488–506
[35] Ince E L 1926 *Ordinary Differential Equations* (London: Longmans Green) (Reprint 1956 (New York: Dover))

[36] Cosgrove C M 1991 Corrections and annotations to E L Ince, *Ordinary Differential Equations*, Chapter 14, on the classification of Painlevé differential equations *Stud. Appl. Math.* submitted

[37] Cosgrove C M 1993 *Stud. Appl. Math.* **89** 1–61

[38] Gambier B 1906 *C. R. Acad. Sci., Paris* **142** 1403–6

[39] Gambier B 1906 *C. R. Acad. Sci., Paris* **142** 1497–1500

[40] Clarkson P A, Fokas A S and Ablowitz M J 1989 *SIAM J. Appl. Math.* **49** 1188–1209

[41] Fuchssteiner B 1993 Some tricks from the symmetry-toolbox for nonlinear equations: the Camassa–Holm equation *University of Colorado preprint* PAM 160